

# An Optimal Mastermind (4,7) Strategy and More Results in the Expected Case

Geoffroy VILLE\*

March 2013

## Abstract

This paper presents an optimal strategy for solving the 4 peg-7 color Mastermind MM(4,7) in the expected case (4.676) along with optimal strategies or upper bounds for other values. The program developed is using a depth-first branch and bound algorithm relying on tight upper bound, dynamic lower bound evaluation and guess equivalence to prune symmetric tree branches.

## 1 Introduction

Mastermind is a code-breaking game for two players. This peg game was invented in 1970 by Mordecai Meirowitz. One of the players – the *code-maker* – chooses a secret code of four pegs of six possible repeatable colors, thus one code among 1296 possibilities. The other player – the *code-breaker* – tries to break the code by making guesses, i.e., submitting one code. The code-maker answers using four pegs of two colors: a black peg means that a guess peg matches both color and position of a code peg, whereas a white peg means that a guess peg matches the color but not the position of a code peg. An answer containing less than four pegs indicates that one or more guess pegs are not part of the code. The answer is global so the code-breaker does not know which black/white peg corresponds to which guess peg. The code-breaker has 8 to 12 guesses (depending of the board) to find the correct code.

The classic board game MM(4,6) can also be played with a virtual seventh color by leaving one or several holes in the code, MM(4,7). It can be generalized to any number of pegs or colors, MM(p,c).

Much research has been done on this game and its variants. Knuth [10] proposed a strategy that requires 4.478 guesses, on average, to find the code (expected case) while always finding the code in a maximum of 5 guesses (worst case). While 5 guesses is the optimal (minimal worst case), several authors proposed different other one-step-ahead heuristic algorithms to reduce the average number of cases until Koyama and Lai [12] found the optimal strategy for the expected case to be 4.340 using a depth-first backtracking algorithm. This optimal strategy requires 6 guesses. Some authors also solved theoretically a few expected cases (Chen and Lin [3], Goddard [5]) or worst cases (Jäger and Peczarski [9]).

---

\* < gville. mastermind at gmail. com >

Higher cases were soon tackled through different approaches like genetic algorithms (Merelo-Guervós et al. [14]) and upper bounds of the expected case were found.

The game of Bulls and Cows, which may date back a century or more and does not allow color repetition, is also worthy of mention as some Mastermind research started on this problem (Chen and Lin [4]).

Another variant called Static Mastermind is also popular. In this game, all the guesses are made at once. Then, given all the answers, the code can be deduced with one additional guess. Minimizing this number of guesses also led to much research (e.g., Goddard [6] or Bogomolny and Greenwell [2]).

To pursue the optimal strategy track in the expected case and to take the list of results published by Goddard [5] a step further, a depth-first branch and bound algorithm with a tight upper bound at the start, a dynamic lower bound evaluation during the resolution, and detecting guess-equivalence at each step using the symmetries of the colors that were played was developed.

Section 2 defines mathematical notations and ways of presenting the problem. Section 3 reviews in more details past work on both MM(4,6) and MM(5,8) and presents or reproduces results of detailed heuristic models. Section 4 describes theoretical established results. Section 5 explains in detail ways to reduce computing time of the program used to find MM(4,7). Section 6 presents results on the expected value. The conclusion makes suggestion for further research.

## 2 Definitions

### 2.1 Mathematical notations

Throughout this article, the following mathematical notations will be used:

$$c = \# \text{ colors} \quad (1)$$

$$p = \# \text{ pegs} \quad (2)$$

$$N_{p,c} = \# \text{ possible codes with } p \text{ pegs and } c \text{ colors} = c^p \quad (3)$$

$$G_p = \# \text{ grades with } p \text{ pegs (and } c \text{ colors)} = \frac{p(p+3)}{2} \quad (4)$$

$$E(p, c) = \text{best average in the expected case for MM}(p, c) \quad (5)$$

$$W(p, c) = \text{best worst case for MM}(p, c) \quad (6)$$

$$MM^*(p, c) = \text{MM}(p, c) \text{ with possible guesses only} \quad (7)$$

While Equation 3 is obvious, Equation 4 requires more explanation. It can be obtained two ways.

The first one is inferred from Table 1. All grading possibilities are on the upper left part of the table. Moreover, on the diagonal, all gradings are possible but  $(p-1, 1)$  for obvious reasons. Thus,

$$G_p = \frac{\overbrace{(p+1) \times (p+1)}^{\text{whole table}} - \overbrace{(p+1)}^{\text{diagonal}}}{2} + p = \frac{(p+1)p + 2p}{2} = \frac{p(p+3)}{2}$$

Table 1: The 14 possible grades in MM(4,7)

$\begin{smallmatrix} w \\ b \end{smallmatrix}$	0	1	2	3	4
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)
1	(1,0)	(1,1)	(1,2)	(1,3)	x
2	(2,0)	(2,1)	(2,2)	x	x
3	(3,0)	x	x	x	x
4	(4,0)	x	x	x	x

The second way comes from the fact that  $G_p$  can also be seen as the number of ways to solve  $b + w + z = p$  ( $z$  integer), minus the impossible solution of  $(b, w) = (p - 1, 1)$ . Therefore,

$$G_p = \binom{p+3-1}{3-1} - 1 = \frac{(p+2)!}{2!p!} - 1 = \frac{(p+1)(p+2) - 2}{2} = \frac{p^2 + 3p}{2} = \frac{p(p+3)}{2}$$

In the MM(4,7) case,  $N_{4,7} = 2401$  and  $G_4 = 14$ .

## 2.2 Summing by colors

The total number of codes can also be thought of as the sum of all the number of possible combinations of  $i$  colors.

Let  $C_i$  be the number of possible codes of  $p$  pegs of exactly  $i$  colors chosen among  $c$ .  $C_i$  is the way of choosing  $i$  colors among  $c$  multiplied by the number of ways of setting these  $i$  colors in a code of length  $p$ , denoted  $Z_{pi}$ . As there are no more than  $p$  colors and  $c$  possibilities,

$$\forall(p, c), N_{p,c} = c^p = \sum_{i=1}^{\min(c,p)} C_i = \sum_{i=1}^{\min(c,p)} \binom{c}{i} Z_{pi}$$

where  $Z_{pi}$  is the sum of all possible distributions of the  $i$  colors and given by the following formula:

$$Z_{pi} = \sum_{n_1+n_2+\dots+n_i=p} \frac{(n_1+n_2+\dots+n_i)!}{n_1!n_2!\dots n_i!} = \sum_{n_1+n_2+\dots+n_i=p} \frac{p!}{n_1!n_2!\dots n_i!}$$

$Z_{p1} = 1$  because there is only one way of placing  $p$  pegs of the same color and,  $\forall p \leq c, Z_{pp} = p!$  because it is the number of ways of coding  $p$  pegs of at least  $p$  colors.

For example, in the MM(4,7) game,

$$\begin{aligned} N_{4,7} &= \binom{7}{1}1 + \binom{7}{2}\left(\frac{4!}{3!1!} + \frac{4!}{2!2!} + \frac{4!}{1!3!}\right) + \binom{7}{3}\left(3 \times \frac{4!}{1!2!1!}\right) + \binom{7}{4}4! \\ &= 7 + 294 + 1260 + 840 \\ &= 2401 \end{aligned}$$

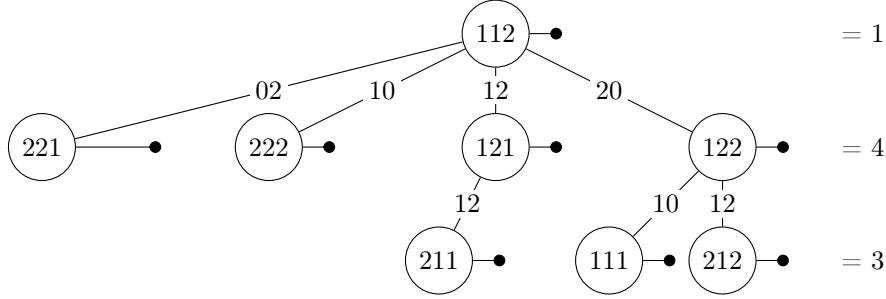


Figure 1: A MM(3,2) optimal tree and the number of codes found at each level

### 2.3 Modeling a solution as a tree

Let's picture a tree representation of a solution (Figure 1). Each node is a guess and each branch an answer. A  $(p, 0)$  branch leads to an end node, also called a leaf node. Any given solution tree has two attributes, its depth<sup>1</sup>  $D$  and its external path length<sup>2</sup>  $L$ .

To compare different strategies, a uniform distribution of codes is assumed. Consequently, the two complementary ways of measuring a strategy performance are the maximum number of guesses needed in the *worst case*  $W$ , and the average number of guesses in the *expected case*  $E$ . They can be expressed in terms of  $D$  and  $L$  as follows:

$$W = D + 1 \quad (8)$$

$$E = \frac{L}{N} \quad (9)$$

In Figure 1,  $D = 2$  and  $L = 1 \times 1 + 2 \times 4 + 3 \times 3 = 18$ , thus  $W = 3$  and  $E = \frac{18}{8} = 2.250$ . More generally, if  $f_i$  is the number of codes *found* at guess  $i$ , then  $N = \sum_{i=1}^w f_i$  and  $L = \sum_{i=1}^w i f_i$ .

Minimizing  $W$  and minimizing  $E$  are two different goals. For example, the optimal value of  $E$  for MM(4,6) is  $E(4,6) = 5625/1296 = 4.340$ , with 6 moves in the worst case[12]. But Knuth [10] showed that one could always find the answer in  $5 < 6$  questions ( $W(4,6) = 5$  in fact) with an increase in the number of moves.

Finally, as both an artistic representation and an illustration of the unbalanced nature of a solution tree, Figure 2 depicts an optimal solution of MM(3,6) in the form of a circular tree or flower.

### 2.4 A peculiar grading function

Let  $g$  be the grading function.  $g$  is commutative, i.e., given 2 codes  $a$  and  $b$ ,  $g(a,b) = g(b,a)$ . Therefore, if a guess  $a$  is graded (2,0) for example, the solution lies among the codes graded (2,0) with  $a$ .

All Mastermind algorithms are based on this fundamental property.

<sup>1</sup>A tree with a single node has a depth of 0.

<sup>2</sup>The external path length is the sum of the path lengths to each leaf node.

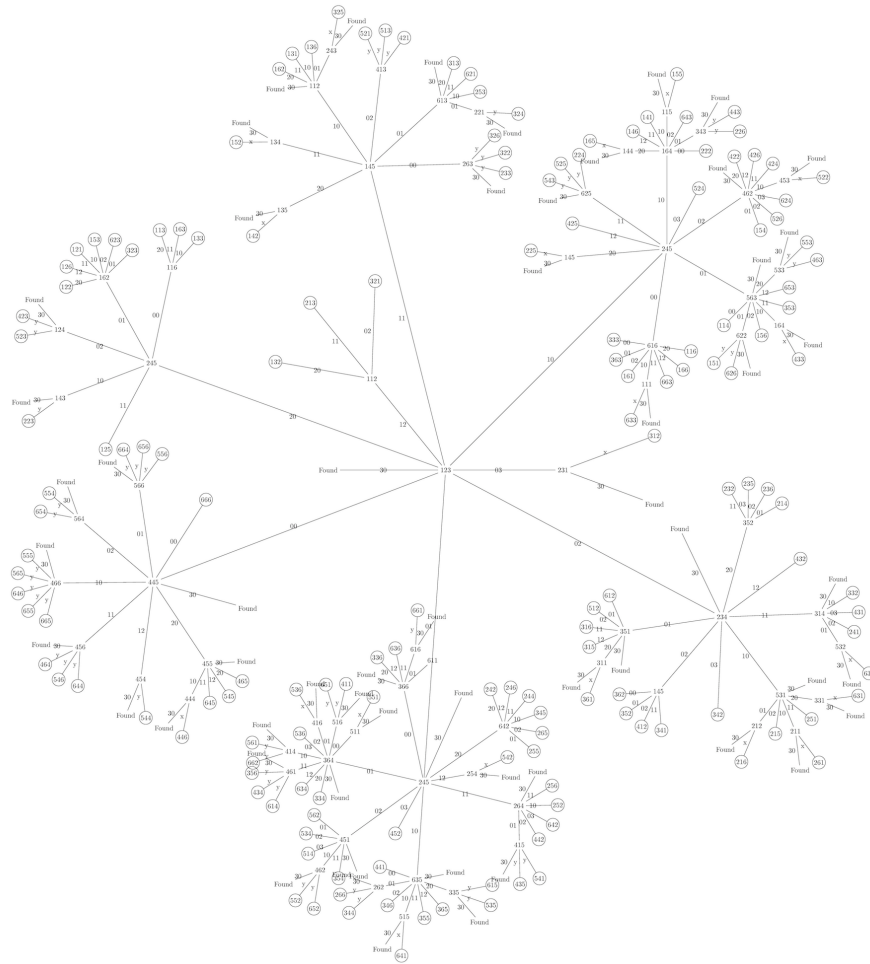


Figure 2: Full ‘flower’ of solutions for MM(3,6)

Table 2: How a non possible code can segregate a set of possible ones

c	121	122	126	153	323	623	<b>162</b>
121	(3,0)	(2,0)	(2,0)	(1,0)	(1,0)	(1,0)	(1,1)
122	(2,0)	(3,0)	(2,0)	(1,0)	(1,0)	(1,0)	(2,0)
126	(2,0)	(2,0)	(3,0)	(1,0)	(1,0)	(1,1)	(1,2)
153	(1,0)	(1,0)	(1,0)	(3,0)	(1,0)	(1,0)	(1,0)
323	(1,0)	(1,0)	(1,0)	(1,0)	(3,0)	(2,0)	(0,1)
623	(1,0)	(1,0)	(1,1)	(1,0)	(2,0)	(3,0)	(0,2)
Diff. grades	3	3	4	2	3	4	<b>6</b>

### 3 Solving Mastermind through heuristics

#### 3.1 Guessing only possible codes is not optimal

Knuth was the first to show that using guesses that are not valid codes when played could reduce the solution tree depth.

This remarkable result is not intuitive and is illustrated by Table 2. It shows a case in which 6 codes are left to be found, and guessing a seventh discriminates all the codes in one question ( $L = 1 \times 0 + 2 \times 6 = 12$ ) whereas playing any of the 6 codes requires at least two ( $L = 1 \times 1 + 2 \times 3 + 3 \times 2 = 13$  in the case of 4 different answers).

For example, when solving MM(4,6) using Knuth's algorithm but with *possible* guesses only, namely  $MM^*(4,6)^3$ , the result goes up to  $5828/1296 = 4.497$ ; after 2 guesses only, 7 (1+6) codes are found while the original algorithm reveals 14 (1+13) codes after two guesses.

Therefore and unfortunately, the number of codes to try at each step is not the monotonously decreasing set that one would hope for, it remains constant minus the previous guesses. Solving the problem requires memory and processor time. Some well-known heuristics have been developed to this end.

#### 3.2 Simple consistency algorithm

The first algorithm one might think of programming is extremely simple yet efficient. It starts with the first code 1111 then:

1. submits a guess and stores the result,
2. chooses, in lexical order, the next guess compatible with the stored results, and
3. stops when  $(b, w) = (p, 0)$ .

On the one hand, its main advantages are simplicity and very low computation and memory cost. Only previous guesses and grades are stored. Each new code is created along the way and graded against the codes in history. There is no need to store the set of possible solutions at each step. On the other hand, since it only uses possible codes and does not try to aim for any optimization, its value is very high. When starting with 1111, the expected case is

<sup>3</sup>We will see in subsection 5.2 how solving a possible case can still be useful.

Table 3: One-step-ahead algorithm results for MM(4,6) vs. the optimal

Algorithm	First Guess	L	E	W
Max. size	1122	5,801	4.476	<b>5</b>
Expect. size	1123	5,696	4.395	6
Entropy	1234	5,723	4.416	6
Most parts	1123	5,668	4.373	6
Optimal	1123	<b>5,625</b>	<b>4.340</b>	6

$7471/1296 = 5.765$ , with 9 guesses in the worst case. Shapiro [17]<sup>4</sup> described this strategy.

Unlike the algorithms described in subsection 3.3, it chooses the next possible solution solely on the lexical order basis. Thus, if the first guess makes the program jump faster toward a suitable reduced range of possible codes, the result improves. Swaszek [18]<sup>5</sup> tried choosing the guesses randomly and found an expected case of 4.638.

A non random approach can also improve the result. The first possible code in lexical order is still chosen at each step. If 1122<sup>6</sup> is forced as the first guess, the expected case reduces to  $6508/1296 = 5.021$  with 8 guesses. If the first guess is 3456, giving information about the colors that are usually explored at the very end, the result becomes  $6045/1296 = 4.664$  and 7 guesses.

As shown in the next section, there is no best first code: depending on which algorithm is used, one code suits it best.

### 3.3 One-step-ahead heuristics

Other algorithms look one step ahead and choose the next guess accordingly. All these algorithms sort, for all guesses, the remaining possible solutions by grade into subsets. They differ by the way, based on these subsets, they select their next guess. When several guesses are eligible, the final choice is made by selecting the possible ones, if any, among them and ultimately choosing the first in lexical order.

Limitations are twofold. Obviously, starting with the best first-step guess only optimizes the next step, while another might have optimized the next two steps and be globally better. Using another sorting policy than lexical order among the eligible guesses at each step has an impact on the results which shows that guesses are not all equivalent (impact on the following steps).

Knuth [10] showed that a guess that would minimize the maximum number of codes in each of  $k$  subsets (*Max. size* algorithm, minimize  $\max(n_i)_{i=1..k}$ ) has an average of  $5801/1296 = 4.476$ . The best first code (lexical order, one step ahead) for this algorithm is 1122.

The next guess can also be selected by minimizing the expected size of the  $k$  subsets (Irving [8]<sup>7</sup>, minimize  $\sum_1^k \frac{n_i^2}{N}$ ), maximizing their entropy (Neuwirth

<sup>4</sup>I did not have access to that paper.

<sup>5</sup>I did not have access to that article either and quote it through what other articles describe.

<sup>6</sup>1122 is among the ‘best’ first guess of Table 3.

<sup>7</sup>Irving used this strategy for the first two guesses and did an exhaustive search after. The pure strategy was used here for all the steps.

Table 4: One-step-ahead algorithm results for MM(4,7)

Algorithm	First Guess	L	E	W
Consistency	4567	12,265	5.108	8
Max. size	1234	11,613	4.837	6
Expect. size	1234	11,409	4.752	6
Entropy	1234	<b>11,382</b>	<b>4.740</b>	6
Most parts	1123	11,388	4.743	6

Table 5: One-step-ahead algorithm results for MM(5,8)

Algorithm	First Guess	L	E	W
Consistency	45678	195,633	5.970	10
Max. size	11234	183,966	5.614	<b>7</b>
Expect. size	11234	180,287	5.502	<b>7</b>
Entropy	11234	179,879	<b>5.489</b>	8
Most parts	11223	181,834	5.549	9

[15]<sup>8</sup>, maximize  $\sum_1^k -p_i \log_2(p_i)$  with  $p_i = n_i/N$ ) or maximizing their number (Kooi [11], maximize  $k$ ).

Table 3 summarizes the results for MM(4,6). *Most parts* gives the lowest score in expected value ( $5668/1296 = 4.373$ ) while *Max. size* gives the lowest number of guesses in the worst case (5). Recreating this table, the same results as [11] were found except for *Entropy* (5723 versus 5722<sup>9</sup>).

The results of these algorithms on the MM(4,7) case are presented in Table 4. This time, *Entropy* performs better. As for MM(4,6), different results are obtained when starting with different first guesses or choosing an order other than lexical.

The results of these algorithms for MM(5,8) were compared with the results of Heeffer and Heeffer [7]. The standard deviation algorithm introduced in the paper was not implemented because of its poor performance. The results found, presented in Table 5, are very different. The result of *Consistency* cannot be compared with the random runs performed by the authors. But for *Max. size*, 5.614 was found versus 5.670, for *Expect. size* 5.502 versus 5.601 and for *Entropy* 5.489 versus 5.583. For *Most parts*, 11223 was used as the first guess (first in lexical order) rather than 11234 reported in the article. Still, the difference is significant (5.549 versus 5.693). The source of the discrepancies was not found.<sup>10</sup> Even though these differences, the algorithms perform in the same order: *Entropy* is the winner, followed by *Expect. size*, *Most parts*, *Max. size* and finally *Consistency*. In terms of worst case, *Entropy* has one code that requires 8 guesses. There should be a way to find 7 guesses by increasing the expected average.

As (p,c) increases, the solution tree size prevents the use of these algorithms and other methods such as genetic algorithm come into play. Merelo and al

<sup>8</sup>I did not have access to his article. The pure strategy was kept.

<sup>9</sup>When running the program taking the last code in lexical order, 5722 is found.

<sup>10</sup>All the MM(4,6) results are quoted in the article and supposedly reproduced by the authors. Furthermore, for MM(5,8), the table of partitions and choice for the first guess are the same. The authors could not be reached.



(e.g., [14] or [13]) or Berghman et al. [1] produced results for higher cases. For MM(5,8), to my knowledge, the best upper bound found in the literature is 5.618 in [1], above *Entropy* 5.489 in Table 5, but reached with a much lower computation time.

## 4 Optimal strategy: some general cases solved

This section presents theoretical results already established. MM(p,1) and MM(1,c) are obvious. MM(2,c) is solved for the general case. MM(3,c) is solved for the pessimistic case.

### 4.1 MM(p,1)

As there is only one code  $\overbrace{1 \dots 1}^p$ ,

$$\forall p, \quad \begin{cases} E(p, 1) &= 1 \\ W(p, 1) &= 1 \end{cases} \quad (10)$$

### 4.2 MM(1,c)

If  $c = 1$ , MM(1,c) is equivalent to the previous case and takes one guess to solve. If  $c = 2$ , it takes two guesses ( $1=1(1,0)$  and  $2=1(0,0)-2(1,0)$ ). So  $E = \frac{1+2}{2} = \frac{3}{2}$  and  $W = 2$ .

Similarly, for  $c = n$ , it always takes  $n$  guesses. Thus,  $W = n$  and

$$E = \frac{1}{n} \times \sum_{i=1}^n i = \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{n+1}{2}$$

Finally,

$$\forall c, \quad \begin{cases} E(1, c) &= \frac{c+1}{2} \\ W(1, c) &= c \end{cases} \quad (11)$$

### 4.3 MM(2,c)

Chen and Lin [3] and Goddard [5] solved this model for  $E$  and  $W$  and obtained the following results:

$$E(2, 2) = 2, \text{ and } \forall c \geq 3, E(2, c) = \begin{cases} \frac{8n^3 + 51n^2 - 74n + 48}{24n^2}, & n \text{ even} \\ \frac{8n^3 + 51n^2 - 80n + 69}{24n^2}, & n \text{ odd} \end{cases} \quad (12)$$

$$\forall c \geq 2, W(2, c) = \lfloor c/2 \rfloor + 2 \quad (13)$$

The results for  $c = 2, \dots, 9$ , presented in Tables 6 and 7, verify these formulas for  $E$  and  $W$ .

As Jäger and Peczarski [9] already noticed, all results for  $W$ , presented in Goddard [5], do not all satisfy Equation 13.

#### 4.4 Partial MM(3,c)

Jäger and Peczarski [9] proved that:

$$\forall c \geq 5, W(3, c) = \lfloor (c-1)/3 \rfloor + 4 \quad (14)$$

and computed  $W(3, c)_{c=2,3,4} = (3, 4, 4)$ .

Once again, the results for  $c = 2, \dots, 9$ , presented in Tables 6 and 7, verify this formula for  $W$ .

This article shows more results for  $W$  and introduces other theoretical inequalities in the pessimistic case.

I am unaware of any paper on  $E(3, c)$ .

### 5 Reaching for MM(4,7)

For higher cases, one must be...patient. The goal was the optimal in the expected case, so the rest of this article will focus on more  $E(p, c)$  results for certain values of  $(p, c)$ .

A depth-first branch and bound algorithm, similar in principle to the depth-first backtracking algorithm used by Koyama and Lai [12] for MM(4,6), was developed. It also explores systematically all possible combinations by branch and goes to the lowest level<sup>11</sup> but saves the best value found (incumbent). As soon as the current branch reaches a higher score, the exploration stops and a new branch is explored. A lower bound evaluation with dynamic update is also used to stop a branch exploration before it actually reaches a higher score: reaching a higher predictive score is sufficient. A tight upper bound comes as a proper incumbent and allows efficient pruning from the start. Finally, some symmetries of the problem are also detected to avoid exploring all codes at each levels.

Given the problem size, optimizing computation time and memory space is crucial. Three main ideas guided this optimization:

1. Use as many shortcuts as possible,
2. Prune the tree as much as possible using a good upper bound from the start and a dynamic lower bound evaluation,
3. Use symmetry as much as possible to avoid exploring similar branches.

#### 5.1 Shortcuts

Some obvious cases encountered along the resolution can be solved right away and can shorten the computation time. To manage the program complexity throughout the years of this work, only simple ones were implemented.

For any set of size  $k < G_p$  codes, if one of them can discriminate all of the others, an absolute minimum is found and this guess can be played right away. The external path is increased by  $1 \times 1 + 2 \times (k-1) = 2k-1$  and 2 guesses are needed at most. Otherwise, if a non-possible code can do the same (see Table 2 for such an example), the external path is increased by one more

<sup>11</sup>The best solution of level  $n+1$  is always sought for each combination of level  $n$ .

$(1 \times 0 + 2 \times k = 2k)$  and at most two guesses are still needed. Each time such configurations are found, a shortcut can be taken.

Conversely, when a code creates only one subset, nothing is gained and the program can backtrack immediately.

There exist refinements of the first shortcut. The  $k = 1$  case is trivial (just play the code!) and can be treated upstream separately. The  $k = 2$  case can also be treated upstream after noticing that simply playing one of the codes followed by the second exactly represents the optimum described. The  $k = 3$  case could also be treated separately but has an impact on the number of guesses. Indeed, either one of the three codes can segregate the other two and an optimum is found ( $1 \times 1 + 2 \times 2 = 5$ ), or playing each of the three codes in sequence leads to 6 ( $1 \times 1 + 2 \times 1 + 3 \times 1 = 6$ ) as playing a non-possible code segregating all others would also lead us to ( $1 \times 0 + 2 \times 3 = 6$ ). But in the latter case, 3 guesses are nevertheless needed instead of 2. This  $k = 3$  case was not implemented as such. This issue is addressed in the conclusion.

Another shortcut deals with an answer of (0,0) after the first guess. When solving the possible case for  $MM(p,c)$ , namely  $MM^*(p,c)$ , the (0,0) answers means that none of the  $k$  colors used in the first guess are part of the code and the branch itself is therefore  $MM^*(p,c-k)$ . In this case, when the program is run to obtain a value and not a solution, using previously found results saves a lot of time. A similar scheme could be applied in the general case. Indeed, in the same situation, but using all possible colors, it would be the same as solving the  $MM(p,c-k)$  but with more available colors (one in fact, as explained in subsection 5.4). Saved  $MMe(p,c)$  data (e for *extended*) from previous runs could also be used in a run where the result is desired rather than the solution. This shortcut was implemented for possible solutions and not for the general case. Note that these values are also used for upper bounds, as explained in the next section, and allow easy detection of discrepancies when testing new versions of the program.

Finally, as obvious as it may seem, storing all pairwise gradings into a table saves computing time at each node. As there is a memory/speed trade-off here, an option to turn it off comes in handy.

## 5.2 A tight upper bound from the start

Keeping the *best* value found so far allows branch pruning each time this incumbent is exceeded. This saves a lot of unnecessary computation and is the basis of the algorithm. But the higher the incumbent, the more unnecessary branches are explored. A good upper bound at the beginning accelerates the pruning process from the start.

An excellent upper bound is the optimal value for  $MM^*(p,c)$ . Such computing option was introduced in the program. This version is obviously much faster because the number of codes is reduced at each step rather than remaining constant. The final result, while not optimal in the general case<sup>12</sup>, is already pretty close. For example, in Table 8, the values for  $MM^*(4,6)$  and  $MM^*(4,7)$  are better than those found with the one-step-ahead algorithms. This ‘possible’ version of the program is fed, in turn, an upper bound from a heuristic computation.

---

<sup>12</sup>Note that the results are the same for 9 cases between Table 6 and Table 8.

### 5.3 A dynamic lower bound evaluation

To be efficient, the pruning process requires, for each branch, a lower bound evaluation beforehand and a dynamic update along its resolution. The pruning occurs sooner as any overrun above the incumbent is not only detected but also predicted.

A classic lower bound can be computed imagining that the remaining codes are found using a perfect tree. Such a tree would have a maximum branching factor of  $G_p$  at each internal node and would be perfectly balanced. After  $q$  questions<sup>13</sup>, all the nodes are leaves. For such a tree, let  $L(p, q)$  denotes the external path length and  $T(p, q)$  the total number of leaves. Then,

$$\forall q \geq 1, T(p, q) = \sum_{i=1}^q (G_p - 1)^{i-1} = \sum_{i=0}^{q-1} (G_p - 1)^i \quad (15)$$

One question implies 1 leaf node ( $(p, 0)$  immediately). Two questions imply  $G_p$  branches and  $(p, 0)$  for all  $G_p - 1$  leaves at question 2, thus  $T(p, 2) = 1 + (G_p - 1) = G_p$  is the total number of nodes and  $L(p, 2) = 1 \times 1 + 2 \times (G_p - 1) = 2G_p - 1$ . This idea was explained in the shortcut section.

Given a number of nodes of  $M$ , and if the perfect tree of  $M$  nodes can be found in  $q + 1$  questions, the external path is equal to the path of  $q + 1$  questions minus the total number found in previous questions:

$$\forall q > 0, T(p, q) < M \leq T(p, q + 1) \Rightarrow$$

$$\begin{aligned} L(p, q + 1, M) &= \sum_{i=1}^q i(G_p - 1)^{i-1} + (q + 1)(M - T_q) \\ &= (q + 1)M - [(q + 1)T_q - \sum_{i=0}^{q-1} (G_p - 1)^i (i + 1)] \\ &= (q + 1)M - \sum_{i=0}^{q-1} (G_p - 1)^i (q + 1 - i - 1) \\ &= (q + 1)M - \underbrace{\sum_{i=0}^{q-1} (G_p - 1)^i (q - i)}_S \end{aligned}$$

Let  $S(p, q)$  denote the *Sum* of external path lengths after  $q$  questions in an optimal tree, i.e.,

$$\forall q \geq 1, S(p, q) = \sum_{i=0}^{q-1} (G_p - 1)^i (q - i), \text{ and } S(p, 0) = 0 \quad (16)$$

then, given a  $p$  value, the lower bound equation becomes

$$T_q < M \leq T_{q+1} \Rightarrow L_{q+1} = (q + 1)M - S_q \quad (17)$$

---

<sup>13</sup>Depth is  $q-1$  by definition.

Both  $S$  and  $T$  are related by recursion equations,

$$\begin{aligned}
\forall q \geq 0, S_{q+1} &= \sum_{i=0}^q (G-1)^i (q+1-i) \\
&= 1 \times (q+1) + \sum_{i=1}^q (G-1)^i (q+1-i) \\
&= (q+1) + \sum_{j=0}^{q-1} (G-1)^{j+1} (q+1-j-1) \\
&= (q+1) + (G-1) \sum_{j=0}^{q-1} (G-1)^j (q-j) \\
&\Rightarrow S_{q+1} = (q+1) + (G-1)S_q
\end{aligned} \tag{18}$$

Similarly,

$$T_{q+1} = T_q + (G-1)^q = 1 + (G-1)T_q \tag{19}$$

And the name of  $S$  is justified by finally noticing that,

$$\begin{aligned}
\forall q \geq 0, S_{q+1} &= \sum_{i=0}^q (G-1)^i (q+1-i) \\
&= \sum_{i=0}^q (G-1)^i (q-i) + \sum_{i=0}^q (G-1)^i \\
&\Rightarrow S_{q+1} = S_q + T_{q+1}
\end{aligned} \tag{20}$$

This lower bound is simple and can be computed efficiently (Equation 17) but it does not perform too well on big sets.

To improve its value, the computation for a node is the sum of the lower bounds of each subset for this node. The evaluation for each subset is replaced by its real value whenever known and the comparison with the incumbent is done at each level instead of at the end.

There exists another way of improving the value of the lower bound. Indeed, while a perfect tree based on the maximum branching factor  $G$  can be imagined, when trying all remaining codes against all guesses, most often  $k < G$  subsets are found. From that point on in the branch, no more than  $k$  subsets will be found. So a higher lower bound can be computed with a branching factor of  $k$  instead of  $G$ . The Equation 19 and Equation 20 allow computation and caching of all values from 2 to  $G$  to improve speed.

This makes the computation highly dynamic and the score converges faster to the real value.

## 5.4 An integrated scheme to detect symmetries and use case equivalence

Symmetry is a key element in the resolution. Many articles mention external programs like *Nauty*<sup>14</sup> to reduce the number of codes by exploiting all symmetries of the game. For speed purposes, a trade-off had to be made between refining

<sup>14</sup><http://cs.anu.edu.au/~bdm/nauty/>

the scheme of symmetry detection and the time spent progressing in the resolution even though more branches might be examined. A method based on the properties of colors at each stage is proposed. A full comparison between the two methods has not been made. The method described hereafter could also be a first code filter upon which other mechanism could be implemented.

As can be understood, for the first guess, all codes but a handful need to be tested. For example, in the seven color case, all seven one-color combinations do not need to be played, only one suffices; the other 6 cases are covered by symmetry, replacing the one color by any of the others. All 2-color, 3-color, 4-color,  $\dots \min(p, c)$ -color cases are also covered in this way. At this stage, it is obvious that neither the colors nor the colors order matter as everything is symmetrical. This complete symmetry is lost once one guess is played, but some interesting properties still remain.

At any stage, any color that has never been played before – call it a *free* color – is symmetrical with any other *free* color. For example, in MM(4,7) and when played after a first guess of 1234, the three guesses 1235, 1236 and 1237 form an interchangeable set of codes, as well as {1255, 1266, 1277} or even {1256, 1257, 1265, 1267, 1275, 1276}. Note that all exclusive *free* color codes keep the complete symmetry of the beginning.

*Zero* colors also play a role. Let *zero* color be any color that is not part of the code. For example, if 1223 gets an answer of (0,0), the three combinations 4511, 4512 and 4531 are all equivalent for the next guess. In fact, any *zero* color can be replaced by any other *zero* color and ultimately the same one. As a result, any exclusive *zero* color guess does not provide additional information and can be filtered out.

Another opportunity to identify *zero* colors comes from the case where  $b + w = p$ . All the pegs are of the right color even though some of them are not in the proper order. Therefore all other colors are *zero* colors.

A signature is assigned to each code based on the respective properties of *free* and *zero* colors. The generic<sup>15</sup> signature is equal to the code where any *zero* colors is replaced by the letter 'z' and a *free* color is replaced, in order of appearance, by a letter in alphabetical order. All codes with the same signature are *case equivalent*<sup>16</sup> and only a class representative<sup>17</sup> needs to be tested. Note that for a given code, its signature evolves along a branch according to the *free* and *zero* colors at the given level. The signature mechanism is especially efficient when  $c \geq p$ . Additional simplification is anticipated in the case of codes made solely of *free* and *zero* colors but the required level of effort could not be dedicated to conclude.

For example, for MM(4,7), the program starts with the well known 5 codes<sup>18</sup> (1111, 1112, 1122, 1223 and 1234) identified by this signature method. In the 1123 branch, never more than 361 of the 2400 possible codes are tried, 41 in the case of a (0,0) answer.

<sup>15</sup>For the first guess or exclusive *free* color codes, the order of all colors is also reorganized.

<sup>16</sup>I did not have access to Neuwirth [15] who seems to be the first to have introduced this notion. From the description in other articles, the main idea of a class of codes given the history of guesses is respected.

<sup>17</sup>The first in lexical order is chosen.

<sup>18</sup>These 5 codes are respectively the representatives of 7, 168 (3+1), 126 (2+2), 1260 and 840 codes. Note that  $168 + 126 = 294$ . See subsection 2.2 for details

## 5.5 A documented example

The following simplified MM(3,4) output illustrates how all these pieces work together.

With 3 pegs and 4 colors:

- there are 9 possible ways of grading,
- starting set has 64 possible combs,

The solver will use:

- for first level, the reduced set of 3 combs (111,112,123),
- an upper bound of 207.

Erase 111 (223>=207)

With the first set :

123 = 194

112 = 196

----- 123 (E=194,g=9)

<123,0>=1

444 ->2 ( ) E:194/S:2

<123,30>=1

123 ->1 ( ) E:194/S:3

<123,3>=2

231 ->5 ( ) E:194/S:8

<123,12>=3

Will try 57 combs finally

112 ->9 (8) E:195/S:17

<123,20>=9

Will try 63 combs finally

134 ->30 (26) E:199/S:47

<123,1>=9

Will try 63 combs finally

244 ->29 (26) E:202/S:76

<123,11>=12

Will try 63 combs finally

134 ->40 (38) E:204/S:116

<123,10>=12

Will try 63 combs finally

244 ->41 (38) E:207/S:157

Min reached already (E:207 or S:157 >= 207). Next one.

----- 112 (E=196,g=9)

<112,30>=1

112 ->1 ( ) E:196/S:1

<112,12>=2

121 ->5 ( ) E:196/S:6

<112,2>=5

Symmetry: 33 versus 63

Will try 32 combs finally

232 ->15 (14) E:197/S:21

<112,11>=8

Symmetry: 33 versus 63

```

Will try 33 combs finally
123 ->24 (23)  E:198/S:45
<112,0>=8
Symmetry: 11 versus 63
Will try 11 combs finally
334 ->26 (23)  E:201/S:71
<112,20>=9
Symmetry: 33 versus 63
Will try 33 combs finally
123 ->30 (26)  E:205/S:101
<112,1>=14
Symmetry: 33 versus 63
Will try 33 combs finally
233 ->47 (46)  E:206/S:148
<112,10>=17
Symmetry: 33 versus 63
Will try 33 combs finally
134 ->58 (58)  E:206/S:206
112 = 206
Found it in 206, starting with 112
Average = 3.22
Full path is:
[111] 112->(20)->123->(10)->111->(30)->Found
<The full solution is of no interest>
[444] 112->(0)->334->(10)->444->(30)->Found
1: 1
2: 5
3: 37
4: 21
Sols= 206
-----

```

Among the 64 possible first guesses, only three class representatives are explored at the first level: one with one color (111), one with two colors (112) and one with three colors (123). An upper bound of  $207^{19}$  is already known from solving the possible case.

Immediately, 111 is discarded because its lower bound of 223 is already higher than the incumbent.

123 has a first estimated lower bound of 194. The resolution pursues until it reaches 207. The real score at that point is only 157.

112 is pursued until the end. When the answer (0,0) is given, only 11 codes out of 63 are explored.

Note how the real branch value and the evaluation beforehand (between brackets) are quite close for the small sets presented. It is good practice to check the evaluation against the real value.

---

<sup>19</sup> $206+1$



## 6 Results

All optimal MM(p,c) results are presented in two tables. Table 6 shows the expected path lengths while Table 7 shows the expected averages. Note that since MM(2,c) is solved, the first line of both tables is for validation purposes.

Already known results (Koyama and Lai [12], Goddard [5]) were found together with five others unpublished to my knowledge (in bold). For MM(4,7),  $E(4,7) = 11228/2401 = 4.676$  with a worst case of 6 guesses<sup>20</sup> using 1123 as the first guess. All these results have the same worst case as the ones in Table 1 of Jäger and Peczarski [9] but for MM(4,6) (the famous 6 versus 5 case).

Table 6: Optimal path lengths in the expected case for MM(p,c)

p \ c	2	3	4	5	6	7	8	9
2	8	21	45	81	132	198	284	388
3	18	73	206	451	854	1,474	<b>2,359</b>	<b>3,596</b>
4	44	246	905	2,463	5,625	<b>11,228</b>		
5	97	816	<b>3,954</b>					
6	224	<b>2,649</b>						
7	496							

Table 8 shows the expected path lengths for all optimal results in the possible case. This table contains 8 more upper bonds (in bold). These values may be helpful when comparing with other algorithms or even trying to solve the optimal case. For MM\*(5,6), only one code requires 7 steps. I therefore believe that  $W(5,6) \leq 6$ . MM\*(5,8) is yet to be found to improve the upper bound of 5.489 found in subsection 3.3.

Finally, Table 9 presents the results from solving MMe(p,c) (solving MM(p,c) with more than c colors) at guess 2. Most of these results are extracted from runs in the case where a first guess received a (0,0) answer. These results could be used to save time when solving higher cases. They were corrected<sup>21</sup> for direct comparison with Table 6. Results are the same but for the cells in gray. For these ‘gray’ cases, a *zero* color is used among the guesses, most often in the first one (second in the global resolution) when for the others only *free* colors are

<sup>20</sup>The  $f_i$  are (1/8/78/717/1473/124)

<sup>21</sup>The computation of  $L$  does not start at guess one but at guess 2,  $\sum_{i=1}^w (i+1)f_i = \sum_{i=1}^w if_i + \sum_{i=1}^w f_i = L + N$

Table 7: Optimal results in the expected case ( $E(p, c)$ )

p \ c	2	3	4	5	6	7	8	9
2	2.000	2.333	2.813	3.240	3.667	4.041	4.438	4.790
3	2.250	2.704	3.219	3.608	3.954	4.297	<b>4.607</b>	<b>4.933</b>
4	2.750	3.037	3.535	3.941	4.340	<b>4.676</b>		
5	3.031	3.358	<b>3.861</b>					
6	3.500	<b>3.634</b>						
7	3.875							

Table 8: Optimal path lengths in the expected case for  $MM^*(p,c)$ 

$\begin{smallmatrix} c \\ p \end{smallmatrix}$	2	3	4	5	6	7	8	9
2	8	21	45	81	134	205	299	417
3	18	73	206	455	864	1,503	2,439	3,749
4	44	247	908	2,476	5,660	11,362	<b>20,838</b>	<b>35,426</b>
5	97	824	3,982	<b>13,572</b>	<b>36,920</b>	<b>86,270</b>		
6	225	2,671	<b>17,416</b>	<b>74,140</b>				
7	505	<b>8,817</b>						

used. It seems to indicate that  $MM(p,c)=MMe(p,c)$  whenever  $c \geq p-1$ , i.e., the fewer the colors compared to the number of pegs, the sooner a *zero* color is needed to discriminate the remaining codes. This conjecture has to be assessed on more examples and confirmed theoretically.

Table 9: Optimal path lengths in the expected case for  $MMe(p,c)$ 

$\begin{smallmatrix} c \\ p \end{smallmatrix}$	1	2	3	4	5	6	7	8
2	1	8	21	45	81	132	198	284
3	1	18	73	206	451	854	1,474	2,359
4	1	40	246	905	2,463			
5	1	91	815					
6	1	189	2,646					
7	1	412						

## 7 Conclusion

An optimal  $MM(4,7)$  strategy was found along with other optimal strategies in the expected case. Additional tight upper bounds (optimal in the possible case) for other cases are also presented as well as an upper bound for  $MM(5,8)$ .

The signature scheme is an efficient way of only testing a class representative and reducing the number of codes tried at each step. The dynamic lower bound mechanism also gives good results. The  $MM^*(p,c)$  upper bound is a good starting value.

After many years of pursuing an optimal  $MM(4,7)$  strategy, the program in its present form has reached its limits in terms of speed and memory space. Perl was used to easily test and implement new ideas over the years while managing the complexity of the program. Further results require a faster and memory-optimized language.

Following are a few ideas that have not been implemented but should enhance the resolution.

The code signature when dealing exclusively with *free* and *zero* colors should be studied and may lead to less representatives.

A generic program for both computing and finding the optimal is not the best solution. To illustrate, let's return to the case where 3 codes are left. We

saw that playing each code one by one would lead to the same  $L$ , but with more guesses, than a true discriminating code as in Table 2. A first phase to find the optimal  $L$  followed by a second one to find the solution itself, with a possible lower tree depth, would be globally faster. In the second phase, the first guess is assumed to be the same and the minimal external length is already known.

This would allow the implementation of the (0,0) first answer for the generic case and not only for the possible case. This idea by itself can lead to some theoretical work that would further help understand the mechanism of *zero* and *free* colors.

Following the same idea, I am certain that more results could also be stored and reused in higher cases. This is where external programs could play an important role to detect such cases and cut the tree by solution blocks. The case of  $b + w = p$  at the first step, while marginal, is such an example. Or other cases where  $k$  never-used-before colors are tried and obtain a (0,0) answer even at a second or further guess. The MMe results could then also be applied.

As computer memory grows, classic computer-algorithm optimizations could also be implemented. More intermediate computations could be saved and used along the way and a two-step ahead mechanism could be programmed to go for the best guess first (using the fact that playing **a** then **b** is equivalent to playing **b** and **a** for the rest of the branch). End games could also be introduced for that matter. This overhead is acceptable for bigger size problems.

Such optimizations would lead to a three-step resolution. A good starting upper bound would be computed through any heuristic algorithm. The first step would use this value to run the possible version with all possible optimizations to find a tighter upper bound. The second step, with all codes, all shortcuts and all optimizations would only search for the optimal  $L$ , no solution would be recorded<sup>22</sup> but the first two guesses leading to this value. Finally, once the optimal value is found, a last pass would find a solution by exploring a small search space using the saved guesses. This last one could eventually focus on a low worst case.

I hope these results will help current researchers of this field but also give ideas to newcomers, as Rosu [16] did for me at the start, several years ago.

## References

- [1] L. Berghman, D. Goossens, and R. Leus. Efficient solutions for Mastermind using genetic algorithms. *Computers and Operations Research*, 36(6):1880–1885, 2009.
- [2] Alex Bogomolny and Don Greenwell. Invitation to mastermind. MAA Online, 1996-2013. URL <http://www.cut-the-knot.org/ctk/Mastermind.shtml>.
- [3] Shan-Tai Chen and Shun-Shii Lin. Optimal Algorithms for  $2 \times n$  Mastermind Games – A Graph-Partition Approach. *The Computer Journal*, 47(5):602–611, 2004.
- [4] Shan-Tai Chen and Shun-Shii Lin. Optimal Algorithms for  $2 \times n$  AB

---

<sup>22</sup>By itself, this speeds up the process.

- Games – A Graph-Partition Approach. *Journal of Information Science and Engineering*, 20(1):105–126, 2004.
- [5] W. Goddard. Mastermind revisited. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 51:215–220, 2004. URL <http://www.cs.clemson.edu/~goddard/papers/mastermindRevisited.pdf>.
  - [6] Wayne Goddard. Static mastermind. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 47:225–236, 2003. URL <http://www.cs.clemson.edu/~goddard/papers/staticMastermind.pdf>.
  - [7] Albrecht Heeffer and Harold Heeffer. Near-optimal strategies for the game of Logik. *International Computer Games Association Journal*, 2008. URL <http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.71.9209>.
  - [8] Robert W. Irving. Towards and optimum mastermind strategy. *Journal of Recreational Mathematics*, 11(2):81–87, 1978.
  - [9] Gerold Jäger and Marcin Peczarski. The number of pessimistic guesses in Generalized Mastermind. *Information Processing Letters*, 109(12):635–641, 2009.
  - [10] D.E. Knuth. The computer as Master Mind. *Journal of Recreational Mathematics*, 9(1):1–6, 1976.
  - [11] Barteld Kooi. Yet another mastermind strategy. *International Computer Games Association Journal*, 28(1):13–20, 2005.
  - [12] Kenji Koyama and Tony W. Lai. An optimal Mastermind Strategy. *Journal of Recreational Mathematics*, 25(4):251–256, 1993.
  - [13] Juan Julián Merelo, Antonio Miguel Mora, Thomas Philip Runarsson, and Carlos Cotta. Assessing efficiency of different evolutionary strategies playing mastermind. In Georgios N. Yannakakis and Julian Togelius, editors, *CIG*, pages 38–45. IEEE, 2010.
  - [14] J.J. Merelo-Guervós, P. Castillo, and V.M. Rivas. Finding a needle in a haystack using hints and evolutionary computation: the case of evolutionary MasterMind. *Applied Soft Computing Journal*, 6(2):170–179, 2006.
  - [15] E. Neuwirth. Some strategies for mastermind. *Mathematical Methods of Operations Research*, 26(1):B257–B278, 1982.
  - [16] Radu Rosu. Analysis of the game of mastermind – the  $m^n$  case. Technical report, Undergraduate thesis, North Carolina State University, 1997. URL <http://www.csc.ncsu.edu/academics/undergrad/Reports/rtrosu/index.html>.
  - [17] Ehud Shapiro. Playing mastermind logically. *SIGART Newsletter*, July 1983(85):28–29, 1983.
  - [18] P. F. Swaszek. The mastermind novice. *Journal of Recreational Mathematics*, 30(3):193–198, 2000.